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Edge Decompositions of Hypercubes by Paths and by Cycles

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Abstract

If H is isomorphic to a subgraph of G, we say that H *divides* G if there exist embeddings $\theta_1, \theta_2, \ldots, \theta_k$ of H such that

$$\{\{E(\theta_1(H)), E(\theta_2(H)), \ldots, E(\theta_k(H))\}$$

is a partition of $E(G)$. For purposes of simplification we will often omit the embeddings, saying that we have an edge decomposition by copies of $E(H)$.

Many authors ([1], [2], [3], [8], [9], [12]) have studied this notion for various subgraphs of hypercubes. We continue such a study in this paper.

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1 Introduction and Preliminary Results

Definition 1 *If H is isomorphic to a subgraph of G, we say that H divides G if there exist embeddings $\theta_1, \theta_2, \ldots, \theta_k$ of H such that*

$$\{\{E(\theta_1(H)), E(\theta_2(H)), \ldots, E(\theta_k(H))\}$$

is a partition of $E(G)$.

Ramras [8] has defined a more restrictive concept.

Definition 2 *A fundamental set of edges of a graph G is a subset of $E(G)$ whose translates under some subgroup of the automorphism group of G partition $E(G)$.*

Edge decompositions of graphs by subgraphs has a long history. For example, there is a Steiner triple system of order n if and only if the complete graph K_n has an edge-decomposition by K_3. In 1847 Kirkman [5] proved that for a Steiner triple system to exist it is necessary that $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$. In 1850 he proved the converse hold also [6].

Theorem 1 *A Steiner system of order $n \geq 3$ exists if and only if $n \equiv 1 \pmod 6)$ or $n \equiv 3 \pmod 6$.*

In more modern times (1964) G. Ringel [11] stated the following conjecture, which is still open.

Conjecture 1 *If T is a fixed tree with m edges then K_{2m+1} is edge-decomposable into $2m+1$ copies of T.*

By Q_n we mean the n-dimensional hypercube. We regard its vertex set, $V(Q_n)$, as $\mathcal{P}(\{1, 2, \ldots, n\})$, the set of subsets of $\{1, 2, \ldots, n\}$. Two vertices x and y are considered adjacent (so $\langle x, y \rangle \in E(Q_n)$) if $|\, x \Delta y \,| = 1$, where Δ denotes the symmetric difference of the two subsets x and y. $(V(Q_n), \Delta)$ is isomorphic as a group to $(\mathbb{Z}_2^n, +)$. Occasionally, when convenient, we shall use the vector notation for vertices; thus $\vec{x}$ and $\vec{y}$ are adjacent precisely when they differ in exactly one component. Note that for $k < n, \mathcal{P}(\{1, 2, \ldots, k\}) \subset \mathcal{P}(\{1, 2, \ldots, n\})$ so that $V(Q_k) \subset V(Q_n)$. In fact, from the definition of adjacency, it follows that Q_k is an induced subgraph of Q_n.

Beginning in the early 1980's, interest in hypercubes (and similar hypercube-like networks such as "cube-connected cycles" and "butterfly" networks) increased dramatically with the construction of massively parallel-processing

computers, such as the "Connection Machine" whose architecture is that of the 16-dimensional hypercube, with $2^{16} = 65,536$ processors as the vertices. Problems of routing message packets simultaneously along paths from one processor to another led to an interest in questions of edge decompositions of $E(Q_n)$ by paths. An encyclopedic discussion of this and much more can be found in [7].

In [8] we have shown that if $\mathcal{G}$ is a subgroup of $\mathrm{Aut}(Q_n)$ and for all $g \in \mathcal{G}$, with $g \neq id$ (where id denotes the identity element), $g(E(H)) \cap E(H) = \emptyset$, then there is a packing of these translates of $E(H))$ in Q_n, *i.e.* they are pairwise disjoint. If, in addition, $|\,E(H)\,| \cdot |\,\mathcal{G}\,| = n \cdot 2^{n-1} = |\,E(Q_n)\,|$, then the translates of $E(G)$ by the elements of $\mathcal{G}$ yield an edge decomposition of Q_n. In [8] it is shown that every tree on n edges can be embedded in Q_n as a fundamental set. (This result for edge decompositions was obtained independently by Fink [3]). In [9] this is extended to certain trees and certain cycles on $2n$ edges. Decompositions of Q_n by k-stars are proved for all $k \leq n$ in [2]. Recently, Wagner and Wild [12] have constructed, for each value of n, a tree on 2^{n-1} edges that is a fundamental set for Q_n. The structure of $\mathrm{Aut}(Q_n)$ is discussed in [8]. For each subset A of $\{1, 2, \ldots, n\}$, the complementing automorphism σ_A is defined by $\sigma_A(x) = A\Delta\{x\}$. Another type of automorphism arises from the group of permutations $\mathcal{S}_n$ of $\{1, 2, \ldots, n\}$. For $x = \{x_1, x_2, \ldots, x_m\} \subseteq \{1, 2, \ldots, n\}$ and $\theta \in \mathcal{S}_n$ we denote by $\rho_{\theta(x)}$ the vertex $\{\theta(x_1), \theta(x_2), \ldots, \theta(x_m)\}$. The mapping $\rho_\theta : V(Q_n) \longrightarrow V(Q_n)$ defined in this way is easily seen to belong to $\mathrm{Aut}(Q_n)$. Every automorphism in $\mathrm{Aut}(Q_n)$ can be expressed uniquely in the form $\sigma_A \circ \rho_\theta$, where this notation means that we first apply ρ_θ. Note: $\rho_\theta \circ \sigma_A = \sigma_{\theta(A)} \circ \rho_\theta$.

To avoid ambiguity in what follows we make this definition:

Definition 3 *By P_k, the "k-path", we mean the path with k edges.*

Questions
(1) For which k dividing $n \cdot 2^{n-1}$ does P_k divide Q_n?
(2) For which k dividing $n \cdot 2^{n-1}$ does C_k, the cycle on k edges, divide Q_n?
(3) For those k for which the answer to either (1) or (2) is "yes", is the edge set used in the decomposition a fundamental set for Q_n?

We begin this introductory section with some examples. In later sections we prove a variety of results relating to these questions, and in the final section we summarize our findings.

Example 1

Let T be the 2-star (= the 2-path) contained in Q_3 with center 000, and leaves $100, 010$. Then $\mathcal{G} = \{id, \sigma_{123}, \sigma_1\rho_{(123)}, \sigma_{12}\rho_{(132)}, \sigma_3\rho_{(132)}, \sigma_{23}\rho_{(123)}\}$ is a (cyclic) subgroup of $\mathrm{Aut}(Q_3)$ of order 6, and the 6 translates of T under $\mathcal{G}$ yield an edge decomposition of Q_3. $\square$

Note, however, that $\mathcal{G}$ does not work for the 2-star T', whose center is 000 and whose leaves are 100 and 001. The subgroup which works for this 2-star is $\mathcal{G}' = \{id, \sigma_{123}, \sigma_1\rho_{(132)}, \sigma_{13}\rho_{(123)}, \sigma_2\rho_{(123)}, \sigma_{23}\rho_{(132)}\}$.

Example 2

P_6 does not divide Q_3. For since Q_3 has 12 edges, if P_6 *did* divide Q_3 then Q_3 would have an edge-decomposition consisting of 2 copies of P_6. The degree sequence (in decreasing order) of each P_6 is $2, 2, 2, 2, 2, 1, 1, 0$, whereas Q_3, of course, is 3-regular. Thus the vertex of degree 0 in one P_6 would require a degree of 3 in the other, which is impossible. $\square$

Example 3

P_4 does not divide Q_3. Since P_4 has 4 edges, we would need 3 copies of P_4 for an edge-decomposition of Q_3. The degree sequence of P_4 is $2, 2, 2, 1, 1, 0, 0, 0$. Call the three copies of P_4 $P^{(1)}, P^{(2)}$, and $P^{(3)}$. At each vertex v of Q_3, $\sum_{1\leq i\leq 3} \deg_{P^{(i)}}(v) = 3$. For each v with $\deg_{P^{(1)}}(v) = 0$ or 2, exactly one of $\deg_{P^{(2)}}(v)$ or $\deg_{P^{(3)}}(v)$ is 1. Thus a total of $6 + 2 = 8$ 1's are required. However, each of the 3 copies of P_4 has exactly 2 vertices of degree 1, and so the total number of 1's in the 3×8 array (a_{iv}) formed by the 3 degree sequences is 6. Hence P_4 does not divide Q_3. $\square$

Example 4

The *other* tree on 4 edges *does* divide Q_3. Let T = the 3-star centered at 000 union the edge $< 001, 101 >$. Let $\mathcal{G} =< \sigma_{23}\rho_{(123)} >$, which is a cyclic subgroup of $\mathrm{Aut}(Q_3)$ of order 3. A straight-forward calculation shows that the translates of T under $\mathcal{G}$ form an edge decomposition of Q_3. $\square$

Proposition 1 *For $k \geq 3, P_{2^k}$ does not divide Q_{2k+1}.*

Proof. Suppose that $k \geq 3$, and suppose that P_{2^k} divides Q_{2k+1}. The matrix (a_{iv}) formed by the degree sequences of copies of P_{2^k} has 2^{2k+1} columns, and

$$(2k+1) \cdot 2^{2k}/2^k = (2k+1)2^k$$

rows. Then since each row has exactly two 1's, the entire matrix has $(2k+1)2^{k+1}$ 1's. But since each vertex of Q_{2k+1} has degree $2k+1$, each column sum is $2k+1$, and thus each column has at least one 1. Thus there must be at least 2^{2k+1} 1's in the matrix. Therefore, $(2k+1)2^{k+1} \geq 2^{2k+1}$. This is equivalent to $2k+1 \geq 2^k$. But for $k \geq 3$ this is clearly false. Thus for $k \geq 3, P_{2^k}$ does not divide Q_{2k+1}. □

We will prove in Section 3 that for $k = 2, P_{2^k}$ *does* divide Q_{2k+1}.

The next result is Proposition 8 of [9].

Proposition 2 *Let n be odd, and suppose that P_k divides Q_n. Then $k \leq n$.*

Lemma 1 *"Divisibility" is transitive,* i.e. *if G_1 divides G_2 and G_2 divides G_3, then G_1 divides G_3.*

Proof. This follow immediately from the definition of "divides". □

Corollary 1 *If k divides n then P_k divides Q_n.*

Proof. By [8], Theorem 2.3, T divides Q_n for *every* tree T on n edges. In particular, then, P_n divides Q_n. Clearly, if k divides n then P_k divides P_n. Hence, by Lemma 1, P_k divides Q_n. □

We have the following partial converse.

Proposition 3 *If P_k divides Q_n and k is odd, then k divides n.*

Proof. Since P_k divides Q_n, k divides $n \cdot 2^{n-1}$. But since k is odd, this means that k divides n. □

Proposition 4 *If k divides n then $E(Q_k)$ is a fundamental set for Q_n. In particular, Q_k divides Q_n.*

Proof. Let $n = k \cdot m$. Define the permutation

$$\theta : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$$

by

$$\theta(i) = i + k \bmod n.$$

Let ρ_θ be the automorphism of Q_n defined by

$$\rho_\theta(x) = \theta(x).$$

The order of θ is m and so the order of the group $\langle \rho_\theta \rangle$ is also m. Let $\mathcal{H} = \{\sigma_A \,|\, A \subset \{k+1, k+2, \ldots, n\}\}$. $\mathcal{H}$ is a subgroup of the automorphism group of Q_{n-k}. Finally, let $\mathcal{G}$ be the group generated by $\mathcal{H}$ and ρ_θ, *i.e.*

$$\mathcal{G} = \{\sigma_A \rho_{\theta^j} \,|\, A \subset \{k+1, k+2, \ldots, n\}, 1 \leq j \leq m, \}.$$

$\mathcal{G}$ is a subgroup of $\mathrm{Aut}(Q_n)$ of order $m \cdot 2^{n-k}$. Hence

$$|\,\mathcal{G}\,| \cdot |\,E(Q_k)\,| = m \cdot 2^{n-k}\left(k \cdot 2^{k-1}\right) = n \cdot 2^{n-1} = |\,E(Q_n)\,|.$$

To show that the translates of $E(Q_k)$ under $\mathcal{G}$ yield an edge decomposition of Q_n it remains to show that for all $A \subset \{k+1, k+2, \ldots, n\}$, and for all $1 \leq j \leq m$,

$$\sigma_A \rho_{\theta^j}(E(Q_k)) \cap E(Q_k) = \emptyset \text{ or } E(Q_k).$$

Now if $e = \langle x, y \rangle \in E(Q_k)$, its direction (the unique element in which its endpoints differ) is some i, then $1 \leq i \leq k$. The direction of $\rho_{\theta^j}(e)$ is $\theta^j(i)$. Automorphisms of the type σ_A leave edge directions fixed, so the direction of $\sigma_A \rho_{\theta^j}(e) = \theta^j(i)$. From the definition of θ, $\theta(i) = k+i$. One can then show by induction on j that $\theta^j(i) = jk+i$ for all j. Hence for $1 \leq j \leq m-1$, $\theta^j(i) > k$. Since all edges of Q_k have direction $\leq k$, for $1 \leq j \leq m-1$, $\sigma_A \rho_{\theta^j}(e) \notin E(Q_k)$; thus $\sigma_A \rho_{\theta^j}(E(Q_k)) \cap E(Q_k) = \emptyset$. Now if $j = m$, θ^m equals id, so $\sigma_A \rho_{\theta^m} = \sigma_A$. But if $\sigma_A \neq id$ then $A \neq \emptyset$. Since $x, y \subseteq \{1, \ldots, k\}$, $\sigma_A(x), \sigma_A(y) \nsubseteq \{1, \ldots, k\}$ and so $\sigma_A(e) \notin E(Q_k)$. Hence $E(Q_k) \cap \sigma_A \rho_{\theta^j}(E(Q_k)) = \emptyset$.

Thus $E(Q_k)$ is a fundamental set for $E(Q_n)$. $\square$

The converse to Proposition 4 follows easily from the next lemma.

Lemma 2 *Suppose that the subgraph H of G edge-divides G. If G is n-regular and H is k-regular, then k divides n.*

Proof. Since the copies of $E(H)$ form an edge-partition of $E(G)$, each vertex v of H must belong to exactly n/k copies of H and so k divides n. $\square$

Corollary 2 *If Q_k divides Q_n then k divides n.*

Proof. Since Q_k is k-regular and Q_n is n-regular, this follows immediately from Lemma 2. $\square$

Combining Proposition 4 and Corollary 2 we obtain

Proposition 5 *Q_k divides Q_n if and only if k divides n.*

As an immediate consequence of Lemma 1 and Proposition 4 we have

Corollary 3 *If k divides n and if P_j divides Q_k then P_j divides Q_n.*

We have a more general consequence.

Corollary 4 *If k divides n and T is any tree on k edges, then there is an embedding of T which divides Q_n. In fact, that embedding of T is a fundamental set for Q_n.*

Proof. By [8], Theorem 2.3, by mapping any given vertex of T to $\emptyset$ and assigning distinct labels $1, 2, \ldots, k$ to the edges of T we get a subtree of Q_k isomorphic to T that divides Q_k, in which no two different edges have the same direction. (We will abuse notation slightly and refer to this tree as T.) Hence by Lemma 1 and Proposition 4, T divides Q_n.

For the second assertion, let $\mathcal{G}_1 = \{\sigma_A \rho_{\theta^j} \mid A \subseteq \{k+1, k+2, \ldots, n\}\}$, where θ is defined as in Proposition 4. Note that the order of θ is m, where $mk = n$. Let $\mathcal{G}_2 = \{\sigma_B \mid B \subseteq \{1, 2, \ldots, k\}, \mid B \mid \text{even}\}$. Finally, let $\mathcal{G} \subseteq \operatorname{Aut} Q_n$ be the subgroup generated by $\mathcal{G}_1$ and $\mathcal{G}_2$. If $\sigma_B \in \mathcal{G}_2$ then $\rho_\theta \sigma_B = \sigma_{\theta(B)} \rho_\theta$. Note that $\theta(B) \subseteq \{k+1, k+2, \cdots ; n\}$ and thus $\sigma_{\theta(B)} \rho_\theta \in \mathcal{G}_1$. So for all j with $1 \leq j \leq m-1, A \cap \theta^j(B) = \emptyset$.

$$A \Delta \theta^j(B) \Delta \theta^{j-1}(B) \cdots \Delta \theta(B) = A \cup [\theta^j(B) \Delta \theta^{j-1}(B) \Delta \cdots \Delta \theta(B)].$$

Hence $\sigma_{A \Delta \theta^j(B) \Delta \cdots \Delta \theta(B)} \rho_\theta \in (\Sigma_n, \rho_\theta),$, the subgroup of $\operatorname{Aut}(Q_n)$ generated by ρ_θ and $\Sigma_n = \{\sigma_C \mid C \subseteq \{1, 2, \cdots, n\}\}$, which is a group isomorphic to the power set of $\{1, 2, \ldots, n\}$ under the operation of symmetric difference. First we shall show that the translates of $E = E(T)$ are edge-disjoint. So suppose

that $\sigma_C \rho_{\theta^j}(e) = e'$, where $e, e' \in E(T)$ and $|\,C\,|$ is even. Let the direction of e be i. Then the direction of $\rho_\theta(e)$ is $\theta(i)$. Since translations σ_A preserve the directions of edges, the direction of e' is also i. Thus $\theta^j(i) = i$. Since θ is a product of disjoint m-cycles, this implies that $j \equiv 0 \pmod{m}$, and hence $\theta^j = \theta^m = id$. Thus $\sigma_C(e) = e', e, e' \in T$. But no two distinct edges of T have the same direction in Q_k. So σ_C either fixes both ends of e and is therefore the identity, or else interchanges the ends of e. But then $|\,C\,| = 1$, contradicting the assumption that $|\,C\,|$ is even. So the translates of $E(T)$ are disjoint.

Finally, we must show that every edge $e \in E(Q_n)$ belongs to *some* translate of $E(T)$. Let $e \in E(Q_n)$. Then since $E(T)$ is fundamental for $E(Q_k)$, $e = g'(e')$ for some $e' \in E(T)$, and for some $g' \in \mathcal{G}_2$. Since $E(Q_k)$ is fundamental for Q_n relative to $\mathcal{G}_1$, $e' = g''(e'')$, where $e'' \in T$ and $g'' \in \mathcal{G}_1$. Thus $e = g'g''(e'')$ for some $e'' \in T$. Since $g'g'' \in \mathcal{G}$, the translates of $E(T)$ by elements of $\mathcal{G}$ *do* cover $E(Q_n)$. So $E(T)$ is fundamental for $E(Q_n)$. $\square$

Proposition 6 *If n is even, and $j < n$ then P_{2^j} divides Q_n.*

Proof. It is proved in [1] that the cycle C_{2^n} divides Q_n. The Hamiltonian cycle C_{2^n} is divisible by any path P_q, as long as q divides 2^n and $q < 2^n$. Thus C_{2^n} is divisible by P_{2^j} provided $j < n$. The result now follows from Lemma 1. $\square$

Proposition 7 *If n is even, and C is the $2n$-cycle with initial vertex $\emptyset$, and edge direction sequence $(1, 2, \ldots, n)^2 \stackrel{\text{def}}{\equiv} (1, 2, \ldots, n, 1, 2, \ldots, n)$, then Q_n is edge-decomposed by the copies of C under the action of $\mathcal{G} = \{\sigma_A \,|\, A \subset \{1, 2, \ldots, n-1\}, \,|\,A\,|$ even$\}$. So $E(C)$ is fundamental for Q_n.*

Proof. C consists of the path P, followed by $\sigma_{\{1,2,\ldots,n\}}(P)$, where P is the path with initial vertex $\emptyset$ and edge direction sequence $1, 2, \ldots, n$. Note that for any $B \subseteq \{1, 2, \ldots, n\}$, for any edge $e, \sigma_B(e) = e \Longrightarrow B = \emptyset$ or $|\,B\,| = 1$. Now we shall show that for every subset $A \subset \{1, 2, \ldots, n-1\}$ with $|A|$ even, $\sigma_A(C) \cap C = \emptyset$. It should be noted that these A's form a subgroup of $\mathrm{Aut}(Q_n)$ of order 2^{n-2}. So suppose that $e = \langle x, y \rangle \in C \cap \sigma_A(C)$. Let the direction of e be i. Then the direction of $\sigma_A(e) = i$. If $A \neq \emptyset$, then since $|A|$ is even, $\sigma_A(e) \neq e$. The only other edge in C with direction i is $\sigma_{\{1,2,\ldots,n\}}(e)$. So if $\sigma_A(e) \in C$, then $\sigma_A(e) = \sigma_{\{1,2,\ldots,n\}}(e)$. Therefore $\sigma_A \cdot \sigma_{\{1,2,\ldots,n\}}(e) = e$, *i.e.* $\sigma_{A\Delta\{1,2,\ldots,n\}}(e) = e$. Since A and $\{1, 2, \ldots, n\}$ are

even, so is $A\Delta\{1,2,\ldots,n\} = \overline{A}$. Hence $A\Delta\{1,2,\ldots,n\} = \emptyset$, *i.e.* $A = \{1,2,\ldots,n\}$. But $n \notin A$, so we have a contradiction.

Thus we have a group $\mathcal{G}$ of automorphisms of C of order 2^{n-2}, such that for $g \in \mathcal{G}, g \neq id$, $g(E(C)) \cap E(C) = \emptyset$. Furthermore, since $|E(C)| = 2n$, it follows that $|\mathcal{G}| \cdot |E(C)| = |E(Q_n)|$. Hence by [8], Lemma 1.1, the translates of $E(C)$ via the elements of $\mathcal{G}$ form an edge decomposition of Q_n. □

Corollary 5 *If n is even, $k < n$ and k divides n, then P_{2k} divides Q_n.*

Proof. Since k divides n, $2k$ divides $2n$, and thus since $2k < 2n, P_{2k}$ divides the $2n$-cycle C of Proposition 7. Hence by Proposition 7, P_{2k} divides Q_n. □

Corollary 6 *If n and k are both even and k divides n, and C is the $2k$-cycle with initial vertex $\emptyset$, and edge direction sequence $(1,2,\ldots,k)^2$, then C divides Q_n.*

Proof. By the proposition, C divides Q_k, and by Proposition 4, Q_k divides Q_n. The result now follows from Lemma 1. □

2 P_4 divides Q_5

If k is odd then by Proposition 3 and Lemma 1 P_k divides Q_n and only if k divides n. Thus the smallest value of k for which Question (1) remains open is $k = 4$. Corollary 5 settles the matter in the affirmative when n is even and thus we now only need to consider the case of n odd. Example 3 shows that P_4 does *not* divide Q_3.

In the next two sections we show that for all odd n with $n \geq 5, P_4$ divides Q_n. We first, in this section, prove the result for $n = 5$. The strategy is to find a subgraph G of Q_5, show that G divides Q_5, and then show that P_4 divides G. In the next section we deduce the general case.

We define G as follows (see figure 1). First, some notation. For $b, c \in \{0,1\}, Q_5^{(***bc)}$ denotes the 3-cube induced by the vertices $x_1x_2x_3x_4x_5$ with $x_4 = b$ and $x_5 = c$. If $a \in \{0,1\}$ $Q_5^{(**abc)}$ is the 2-cube induced by the vertices with $x_3 = a, x_4 = b$, and $x_5 = c$. We take G to be the union of (1) : $Q_5^{(***00)}$, with the edges of $Q_5^{(*0*00)}$ deleted; (2): $Q_5^{(***10)}$ with all edges deleted except for $\langle 01010, 01110\rangle$ and $\langle 11010, 11110\rangle$; (3) : $Q_5^{(***01)}$ with all edges deleted except for $\langle 01101, 11101\rangle$ and $\langle 01001, 11001\rangle$; (4): the 4 matching

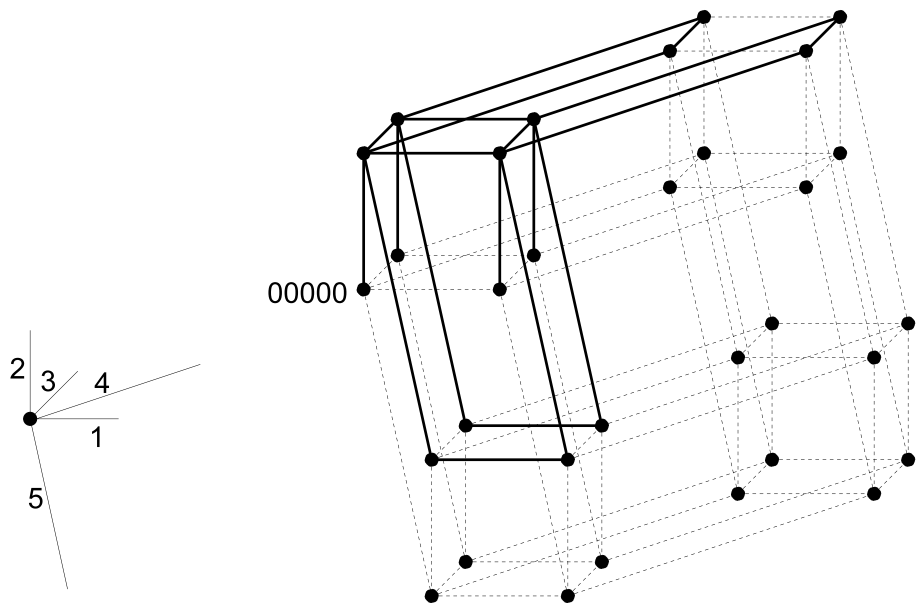

Figure 1: Q_5 and the graph G

edges between $Q_5^{(*1*00)}$ and $Q_5^{(*1*10)}$; and (5) the 4 matching edges between $Q_5^{(*1*00)}$ and $Q_5^{(*1*01)}$. Thus $|\,E(G)\,| = 20$. Since $|\,E(Q_5)\,| = 5 \cdot 2^4 = 80$, we must exhibit $80/20 = 4$ copies of $E(G)$ that partition $E(Q_5)$.

Lemma 3 *G divides Q_5. In fact, $E(G)$ is a fundamental set for Q_5.*

Proof. By direct inspection of figure 2 the group of translations $\mathcal{G} = \{id, \sigma_{24}, \sigma_{25}, \sigma_{45}\}$, applied to $E(G)$, partitions $E(Q_5)$. □

Lemma 4 *P_4 divides G.*

Proof. It is easiest to describe the paths by their starting points and direction sequences (see figure 3).

Path	Starting Point	Direction Sequence
A	00000	$2, 5, 1, 5$
B	10100	$2, 5, 1, 5$
C	10000	$2, 3, 1, 3$
D	01000	$1, 4, 3, 4$
E	00100	$2, 4, 3, 4$

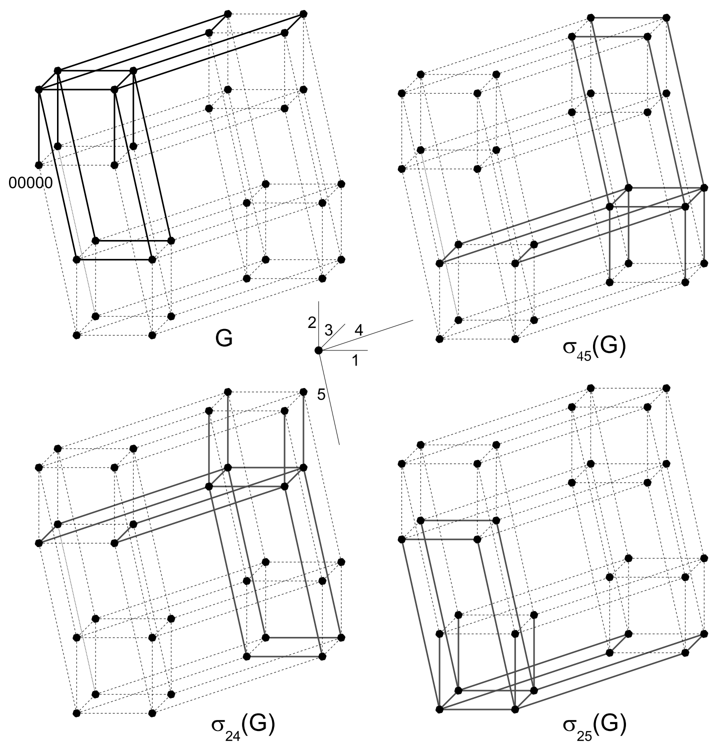

Figure 2: $E(G)$ is a fundamental set for Q_5

$\square$

Corollary 7 *P_4 divides Q_5.*

Proof. This follows immediately from the previous two lemmas. $\square$

3 P_4 divides Q_n, for n odd, $n \geq 5$

Let us write Q_5 as $Q_5 = Q_3 \square Q_2 = Q_3 \square C_4$. Let $G_0 = Q_5^{(***00)}, G_1 = Q_5^{(***10)}, G_2 = Q_5^{(***11)}, G_3 = Q_5^{(***01)}$. For $i \in \{0, 1, 2, 3\}$ let π_i be the canonical mapping from G_i to Q_3.

$*$ From the decomposition of Q_5 by P_4 we have a coloring $c : Q_5 \longrightarrow$

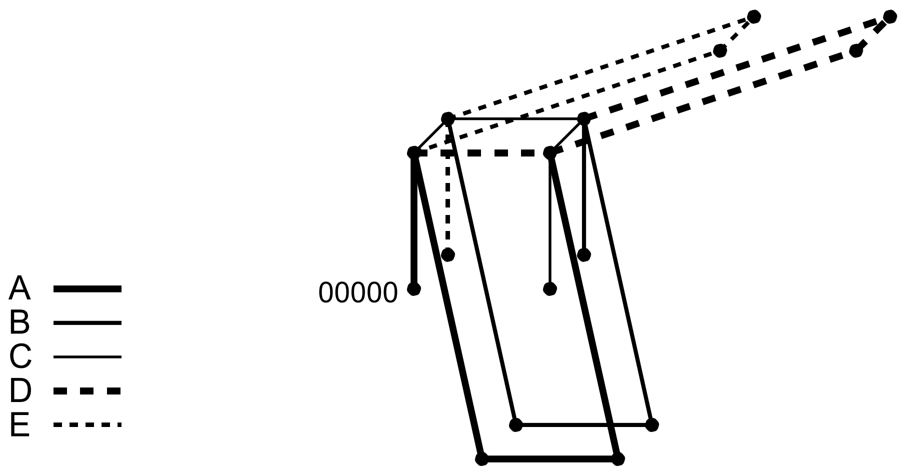

Figure 3: P_4 divides G

$\{1, 2, \ldots, 20\}$ of the edges of Q_5 such that for any $i \in \{1, 2, \ldots, 20\}$ the set of edges of Q_5 colored i induces a P_4.

$*$ Consider now $Q_3 \Box C_{4k}$ for some $k \geq 1$. Let $G'_0, \ldots, G'_{4k-1} \simeq Q_3$. Let $\pi'_{i'}$ be the canonical mapping from $G'_{i'} \longrightarrow Q_3$ for $i' \in \{0, 1, \ldots, 4k-1\}$.
The edges of $Q_3 \Box C_{4k}$ are

Case A: the edges of $G'_{i'}$ for any $i' \in \{0, 1, \ldots, 4k-1\}$.

Case B: for any $i' \in \{0, 1, \ldots, 4k-1\}$ the edges $\langle x', y' \rangle$ for $x' \in G'_{i'}$, $y' \in G'_{j'}$, where $|\, j' - i' \,| \equiv 1 \pmod{4k}$ and $\pi_{i'}(x') = \pi_{j'}(y')$.

$*$ Let θ be the mapping from $Q_3 \Box C_{4k} \longrightarrow Q_5$ defined by: for any $x' \in G'_{i'}, \theta(x') = x$ where x is the element of G_i, with $i \equiv i' \pmod 4$ such that $\pi_i(x) = \pi_{i'}(x')$. (Note that θ is not a one-to-one mapping.)

Proposition 8 *If $\langle x', y' \rangle$ is an edge of $Q_3 \Box C_{4k}$ then $\langle \theta(x'), \theta(y') \rangle$ is an edge of Q_5.*

Proof.

Case A

$\langle x', y' \rangle \in G'_{i'}$ for some i'. Then let $i \equiv i' \pmod 4$. By the definition of $\theta, \theta(x') \in G_i, \theta(y') \in G_i$. This implies that $\theta(x')$ and $\theta(y')$ are adjacent.

Case B

Assume $x' \in G'_{i'}, y' \in G'_{j'}$ with $|\,j' - i'\,| \equiv 1 \pmod{4k}$. We have $\pi'_{i'}(x') = \pi'_{j'}(y\,')$. Then $\theta(x') \in G_i$ and $\theta(y') \in G_j$ where $|j-i| \equiv 1 \pmod 4$ since $|\,j' - i'\,| \equiv 1 \pmod 4$ implies that $|\,j - i\,| \equiv 1 \pmod 4$. Furthermore

$$\pi_i(\theta(x')) \stackrel{\text{def of}\theta}{=} \pi(x') \stackrel{\text{edge}}{=} \pi(y') \stackrel{\text{def of}\theta}{=} \pi_j(\theta(y')).$$

Thus there exists an edge between $\theta(x')$ and $\theta(y')$ $\square$

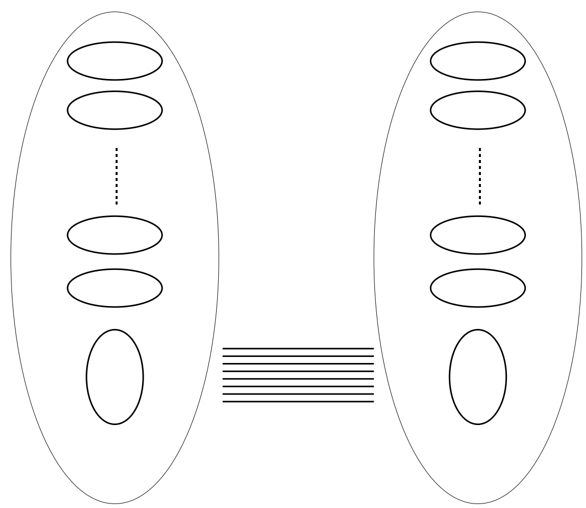

Figure 4: Decomposition of Q_{2m+1}

Definition 4 *Consider the coloring* $E(Q_3 \Box C_{4m}) \xrightarrow{c'} \{1, 2, \ldots, 20\}$ *of the edges of* $Q_3 \Box C_{4m}$ *defined by* $c'(\langle x', y' \rangle) = c(\langle \theta(x), \theta(y) \rangle)$.

Lemma 5 *For any* $i \in \{1, 2, \ldots, 20\}$ *the set of edges of* $Q_3 \Box C_{4k}$ *such that* $c'(x', y') = i$ *is a set of disjoint paths of length* 4. *Therefore* P_4 *divides* $Q_3 \Box C_{4m}$ *for all* $m \geq 1$.

Proof. By definition of c', for any vertex x' of $Q_3 \Box C_{4k}$ the number of edges incident to x' colored i by c' is the number of edges incident to $\theta(x')$ colored i by c. Therefore this number is ≤ 2. Furthermore, there is no cycle colored

i in $Q_3 \square C_{4k}$ because the image by θ of this cycle would be a cycle of Q_5 colored i with c. Therefore the set of edges colored i by c' is a forest and more precisely, because of the degree, a set of disjoint paths.
Notice that the image by θ of a path colored i is a path of Q_5 of the same length (because of the degree of the endpoints of the paths). Therefore all the paths are of length 4. $\square$

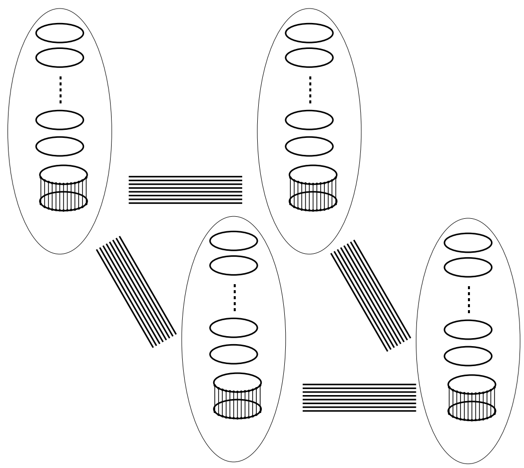

Figure 5: Decomposition of Q_{2m+3}

Theorem 2 *For $n \geq 4$, P_4 divides Q_n.*

Proof. If n is even, the result is true by Corollary 5. If $n = 5$ then we are done by Corollary 7. Consider Q_{2m+3}, for $m \geq 2$. $Q_{2m+3} = Q_{2m+1} \square Q_2$. $E(Q_{2m})$ can be decomposed into m cycles of length 2^{2m} (Hamiltonian cycles) by Aubert and Schneider [1]. Let D be one of these cycles. The edges of Q_{2m+1} are the edges of the two copies of Q_{2m} and a matching. But every vertex of Q_{2m} appears exactly once in D so $E(Q_{2m+1})$ can be decomposed into $2(m-1)$ cycles of length 2^{2m} and $D \square Q_1 \simeq C_{2^{2m}} \square Q_1$ (see figure 4).

Every vertex of Q_{2m+1} appears once in $D \square Q_1$, thus, for the same reason, $E(Q_{2m+3})$ can be decomposed into $8(m-1)$ cycles of length 2^{2k} and $D \square Q_1 \square Q_2 \simeq C_{2^{2k}} \square Q_1 \square Q_2 \simeq C_{2^{2m}} \square Q_3$ (see figure 5).

Since $k \geq 2$, $\frac{2^{2m}}{4}$ is an integer stictly greater than 1 so the cycles of length 2^{2m} are divisible by P_4. By Lemma 5, P_4 divides $C_{2^{2m}} \square Q_3$, and P_4 divides $E(Q_n)$ for any odd $n \geq 5$. $\square$

4 Q_{2^k} has a fundamental Hamiltonian cycle.

Remark. We say that the direction of an edge $\langle x, y\rangle$ of Q_n is i if $x\Delta y = \{i\}$. We shall describe walks in the hypercube by specifying the starting vertex (generally $\emptyset$) and the sequence of edge directions.

It is well-known that the n-dimensional hypercube Q_n is Hamiltonian, and in fact has many Hamiltonian cycles. Aubert and Schneider [1] proved that for n even, Q_n has an edge decomposition into Hamiltonian cycles. However, their construction is technical. In contrast, in this last section we shall prove that for $n = 2^k$, there is a single Hamiltonian cycle C such that $E(C)$ is a fundamental set for Q_n.

By $G_1 \Box G_2$ we denote the Cartesian product of the graphs G_1 and G_2. We will start with two easy results about Cartesian product of graphs.

Lemma 6 *Assume that* $\{C^1, C^2, \ldots, C^p\}$ *is an edge decomposition in Hamiltonian cycles of a graph* G*. Then* $\{C^1 \Box C^1, C^2 \Box C^2, \ldots, C^p \Box C^p\}$ *is an edge decomposition of* $G \Box G$*.*

Proof. Let (x_1, x_2) and (y_1, y_2) be adjacent in $G \Box G$. Then either x_1 and y_1 are adjacent in G and $x_2 = y_2$ or $x_1 = y_1$ and x_2 and y_2 are adjacent in G. By symmetry, it is sufficient to consider the first case. Let i be such that $\langle x_1, y_1\rangle \in E(C^i)$. Then since C^i is Hamiltonian $x_2 = y_2 \in V(C^i)$; thus $\langle (x_1, x_2), (y_1, y_2)\rangle \in E(C^i \Box C^i)$. Conversely $\langle (x_1, x_2), (y_1, y_2)\rangle \in E(C^j \Box C^j)$ implies $\langle x_1, y_1\rangle \in E(C^j)$ since $x_2 = y_2$; thus $j = i$. Therefore the $C^j \Box C^j$'s are disjoint and the conclusion follows. □

Lemma 7 *Let* G_1 *and* G_2 *be any two graphs, and for* $i = 1, 2$ *let* $\phi_i \in \mathrm{Aut}\,(G_i)$*. Define* $(\phi_1, \phi_2) : G_1 \Box G_2 \longrightarrow G_1 \Box G_2$ *by* $(\phi_1, \phi_2)((x, y)) = (\phi_1(x), \phi_2(y))$*. Then* $(\phi_1, \phi_2) \in \mathrm{Aut}\,(G_1 \Box G_2)$*.*

Proof. Let (x_1, x_2) and (y_1, y_2) be adjacent in $G_1 \Box G_2$. Then either (1) x_1 and y_1 are adjacent in G_1 and $x_2 = y_2$ or (2) $x_1 = y_1$ and x_2 and y_2 are adjacent in G_2. We must show that $(\phi_1, \phi_2)(x_1, x_2)$ and $(\phi_1, \phi_2)(y_1, y_2)$ are adjacent in $G_1 \Box G_2$. By symmetry, it is sufficient to prove this for case (1). But then since $\phi_1 \in \mathrm{Aut}\,(G_1)$, $\phi_1(x_1)$ and $\phi_1(y_1)$ are adjacent in G_1, and since $x_2 = y_2$, $\phi_2(x_2) = \phi_2(y_2)$. Therefore $(\phi_1, \phi_2)(x_1, x_2)$ and $(\phi_1, \phi_2)(y_1, y_2)$ are adjacent in $G_1 \Box G_2$. Conversely if $(\phi_1, \phi_2)(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$ and $(\phi_1, \phi_2)(y_1, y_2) = (\phi_1(y_1), \phi_2(y_2))$ are adjacent in $G_1 \Box G_2$ then $\phi_1(x_1) = \phi_1(y_1)$ or $\phi_2(x_2) = \phi_2(y_2)$. We can assume the first case by symmetry then $x_1 = y_1$

and x_2 is adjacent to y_2 in G_2. Thus (x_1, x_2) and (y_1, y_2) are adjacent in $G_1 \square G_2$ and $(\phi_1, \phi_2) \in$ Aut $(G_1 \square G_2)$. $\square$

The starting point of the theorem of Aubert and Schneider is an earlier result of G. Ringel [10] who proved that for $n = 2^k$, Q_n has an edge decomposition into Hamiltonian cycles. His proof is by induction on k. Let us recall the induction step. Let $m = 2^n$. Let θ be the mapping from $\{1, \dots, n\}$ to $\{n+1, \dots, 2n\}$ defined by $\theta(i) = i + n$. Let C be a Hamiltonian cycle of Q_n then we can construct $\Phi(C)$ and $\Gamma(C)$ two disjoint Hamiltonian cycles of $Q_{2n} = Q_n \square Q_n$ such that $E(C \square C) = E(\Phi(C)) \cup E(\Gamma(C))$. Indeed fix an arbitrary vertex (say 0) and represent C by the sequence of directions $C = (c_1, \dots, c_m)$ then consider

$$\Phi(C) = \left(\begin{array}{ll} c_1, \dots & \dots, c_{m-1}, c_{\theta(c_1)}, \\ c_m, c_1, \dots & \dots, c_{m-2}, c_{\theta(c_2)}, \\ c_{m-1}, c_m, c_1, \dots & \dots, c_{m-3}, c_{\theta(c_3)}, \\ \dots\dots & \dots\dots \\ c_2, \dots & \dots, c_m, \; c_{\theta(c_m)}, \end{array} \right)$$

and

$$\Gamma(C) = \left(\begin{array}{ll} c_{\theta(1)}, \dots & \dots, c_{\theta(m-1)}, c_1, \\ c_{\theta(m)}, c_{\theta(1)}, \dots & \dots, c_{\theta(m-2)}, c_2, \\ c_{\theta(m-1)}, c_{\theta(m)}, c_{\theta(1)}, \dots & \dots, c_{\theta(m-3)}, c_3, \\ \dots\dots & \dots\dots \\ c_{\theta(2)}, \dots & \dots, c_{\theta(m)}, \; c_m, \end{array} \right).$$

It is immediate to check (see figure 6) that $\Phi(C)$ and $\Gamma(C)$ are disjoint and define a partition of the edges of $C \square C$. For n even let $p = n/2$ and assume that $\{C^1, C^2, \dots, C^p\}$ is an edge decomposition of Q_n in Hamiltonian cycles then as a consequence of Lemma 6, $\{\Phi(C^1), \Phi(C^2), \dots, \Phi(C^p)\} \cup \{\Gamma(C^1), \Gamma(C^2), \dots, \Gamma(C^p)\}$ is an edge decomposition of Q_{2n} in Hamiltonian cycles.

Theorem 3 *For any $k \geq 1, Q_{2^k}$ has a Hamiltonian cycle that is a fundamental set.*

Proof. This is trivial for $k = 1$ since $Q_2 = C_4$. The desired result follows by induction from Ringel's construction. Indeed let $n = 2^k, k \geq 1$ and assume that there exists an edge decomposition $\{C^1, C^2, \dots, C^p\}$ of Q_n obtained as the translate of an Hamiltonian cycle C^1 under some subgroup $\mathcal{E}$ of Aut (Q_n). For any automorphism $\phi \in$ Aut (Q_n), $(\phi, \phi) \in$Aut (Q_{2n}) by Lemma 7. Furthermore if $\phi(C^1) = C^i$ then $(\phi, \phi)(\Phi(C^1)) = \Phi(C^i)$ and

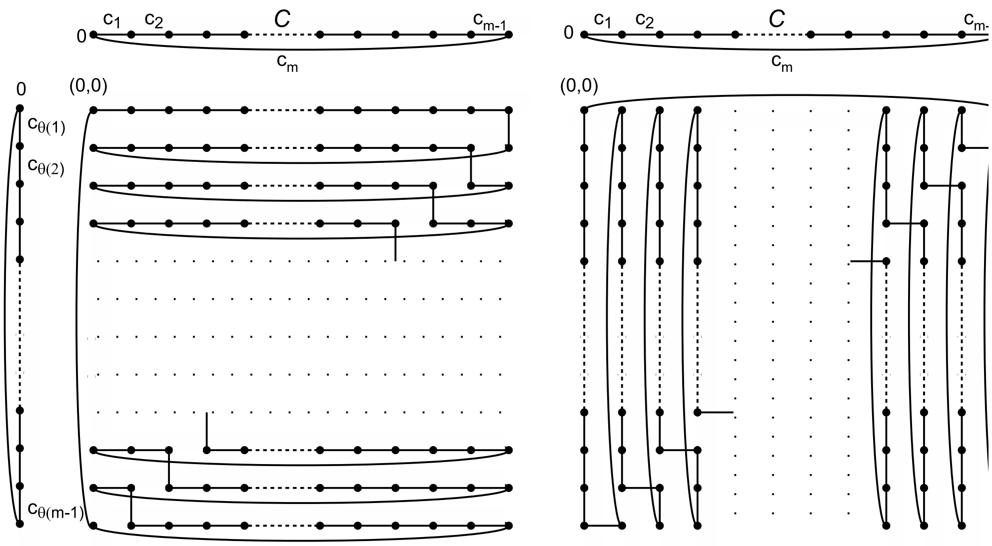

Figure 6: Construction of $\Phi(C)$ and $\Gamma(C)$ from C

$(\phi,\phi)(\Gamma(C^1)) = \Gamma(C^i)$. If we consider now the permutation θ on $\{1,\ldots,2n\}$ defined by $\theta(i) = i + n \; mod \; 2n$ then $\rho_\theta(\Phi(C^i)) = \Gamma(C^i)$. The conclusion follows since the subgroup of Aut (Q_{2n}), isomorph to $\mathcal{E} \times S_2$, defined by $\mathcal{H} = \{(\phi,\phi); \phi \in \mathcal{E}\} \cup \{\rho_\theta \circ (\phi,\phi); \phi \in \mathcal{E}\}$ is such that $\{\Phi(C^1), \Phi(C^2), \ldots, \Phi(C^p)\} \cup \{\Gamma(C^1), \Gamma(C^2), \ldots, \Gamma(C^p)\}$ are the translate of $\Phi(C^1)$ under $\mathcal{H}$. $\square$

Corollary 8 *For n and m each a power of 2, with $m \leq n$, there is an m-cycle that divides Q_n.*

Proof. Let $m = 2^p$. By Theorem 3 Q_m has a fundamental 2^p-cycle, which therefore divides $Q_m = Q_{2^p}$. Since m and n are each powers of two, m divides n. Hence by Proposition 4 and Lemma 1, this cycle divides Q_n. $\square$

5 Summary of Results

1. For k odd, if P_k is a path on k edges that divides Q_n, then k divides n. (Proposition 3)

2. If k divides n, *any* tree on k edges is a fundamental set for Q_n. (Corollary 4)
3. If k divides n and $k < n$ then P_{2k} divides Q_n. (Corollary 5)
4. If n is even and $j < n$ then P_{2^j} divides Q_n. (Proposition 6)
5. For $k = 2n$ there is a k-cycle which is a fundamental set for Q_n when n is even. (Proposition 7)
6. For $n =$ a power of 2, there is a Hamiltonian cycle which is a fundamental set for Q_n. (Theorem 3)
7. For $n =$ a power of 2 and $m =$ a power of 2, with $m \leq n$, there is an m-cycle that divides Q_n. (Corollary 8)
8. For $n \geq 4$, P_4 divides Q_n. (Theorem 2)
9. Q_k is a fundamental set for Q_n if and only if k divides n. (Proposition 5)
10. For $k \geq 3, P_{2^j}$ does *not* divide Q_{2k+1}. (Proposition 1)

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